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Numerical solution of ordinary differential equations in Euler method

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Abstract— **This work presents numerical methods for solving initial value problems in ordinary differential equations. Euler's method is presented from the point of view of the tangent line simplifies the rigorous analysis while Runge-Kutta method attempts to obtain at the same time avoid the need for higher derivatives by evaluating the given function at selected points on each subinterval.**

***Keywords*— Euler's Methods, Modified Euler's Methods, Runge-kutta method, Example and problem.**

I. INTRODUCTION

A differential equation is an equation for a function that relates the values of the function to the values of its derivatives.

An ordinary differential equation (ODE) is a differential equation for a function of a single variable, e.g., (⍰⍰), while a partial

differential equation (PDE) is a differential equation for a function of several variables, e.g., (⍰⍰, ⍰⍰, ⍰⍰, ⍰⍰). An ode contains

ordinary derivatives and ODE contains partial derivatives.

A.The Euler method

Simple differential equations, it is possible to find closed form solutions.

For example, Given a function f, the general solution of the simplest equation,

$$- = (,)$$

is

$$( ) = \int ( ) \quad +$$

Where c an arbitrary integration constant. Here $\int ()$ denotes any fixed anti- derivative of f. The constant c, and thus a particular solution, can be obtained by specifying the value of y (t) at some given point:

y (0) = ▪.

Euler's method is a numerical technique to solve ordinary differential equations of the form

$$-=(,)$$, y (0) = ▪.

So only first order ordinary differential equations can be solved by using Euler's method.

Example

The general solution of the equation

$$-=\sin$$

Is

$$()=-\cos +$$.

If we specify the condition

$$y(/3)=2$$

Then it is easy to find c = 2.5. Thus the desired solution is

$$()=2.5-\cos$$.

The more general equation

$$-=(,) \quad (1.1)$$

This approached in a similar spirit, in the sense that usually there is a general solution Dependent on a constant. To further this point, we consider some more Examples that can be solved analytically. First, and foremost, is the first-order linear Equation

$$'()=()()+() \quad (1.2)$$

The given functions $()$ and $()$ are assumed continuous. For this equation,
We Obtain

f(x,y)=a(x)y + f(x),

And the general solution of the equation can be found by the so-called *method of integrating factors*.
The method of integrating factors through a particularly useful case

$$'()=()+() \quad (1.3)$$

Where a given constant. Multiplying the linear equation (1.3) by the integrating factor $^{-}$, we can reformulate the equation as

$$\frac{(^{-}())}{}=^{-}()$$

Integrating both sides from ▪ to , we obtain

$$^{-}()=\int_{▪}^{}{}^{-}()+$$

$$\text{Where} \quad =^{-▪}(_{▪}) \quad (1.4)$$

So the general solution of (1.3) is

$$(\,)=\ \{\int_{0}[\ -\ \ (\,)\ \ +\ \ ] \qquad (1.5)$$

This solution is valid on any interval on which f(x) is continuous. As we have seen from the discussions above, the general solution of the first-order Equation (1.1) normally depends on an arbitrary integration constant. To single out A particular solution, we need to specify an additional condition. Usually such a Condition is taken to be of the form

$$(_{0})=\ _{0} \qquad (1.6)$$

In many applications of the ordinary differential equation (1.1), the independent variable plays the role of time, and $_0$ can be interpreted as the initial time. So it is customary to call (1.6) an *initial value condition*. The differential equation (1.1) and the initial value condition (1.6) together form an *initial value problem*

$$-=(\,,\,),\ (_{0})=\ _{0} \qquad (1.7)$$

For the initial value problem of the linear equation (1.3), the solution is given by the formulas (1.5) and (1.4). We observe that the solution exists on any open interval where the data function f( ) is continuous. This is a property for linear equations. For the initial value problem of the general linear equation (1.2), its solution exists. On any open interval where the functions a( ) and f( ) are continuous. As we will see next through examples, when the ordinary differential equation (1.1) is nonlinear, Even if the right-side function f(,) has derivatives of any order, the solution of the corresponding initial value problem may exist on only a smaller interval.

A. *Derivation of Euler's method*

At =0, we are given the value of (0) = $_0$. Let us call x=0 as $_0$. Now since we know the slope of y with respect to x, that is f(,) , then at = $_0$, the slope is f($_1 0,$ $_1 0)$. Both $_0$ and $_0$ are known from the initial condition. y($_0$)= $_0$.

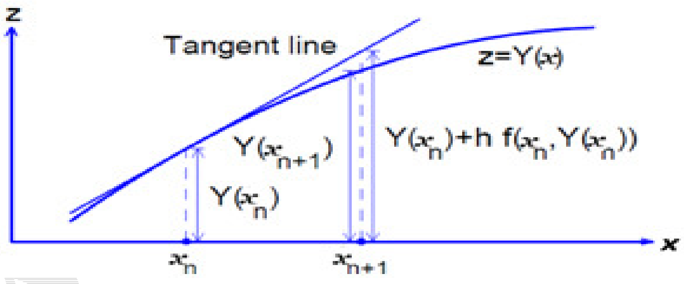


Fig.A *Graphical interpretation of the first step of Euler's method.*

Euler's Method
So the slope at x= $_0$ as shown in Figure 1.1.1 is

$$=\ \frac{}{}$$

$$=\frac{_{1}-\ _{0}}{_{1}-\ _{0}}$$

$$=f(\ _{1}(0,)\ \ _{1}0)$$

From here

$$_{1}=\ _{0}+(\ ,\ )[(\,]\ -\ \ )$$

Calling $_1 -\ _0$ the step size h, we get

$$_1 = _0 + (,)h$$

One can now use the value of $_1$ (an approximate value of at y as x= $_1$) to calculate $_2$ and that would be the predicted value at $_2$ given by

$$_2 = _1 + (,)h.$$

Where $_2 = _1 + h$

Based on the above equations, if we know the value of y= at ,then

$$_{+1} = + hf(,)$$

This formula is known as Euler's method and is illustrated graphically in Fig A. It is also Called the Euler-Cauchy method.

Euler's method based on problems

Using Euler's method, find y(0.2) and y(0.4) from $— = +$, y(0)=1with h=0.2.

Solution:

Euler's Method

So the slope at x= $_0$ as shown in Fig.A is

$$\text{slope}=\frac{\;}{_1 - _0}$$

$$= _1 - _0$$

$$=f(_1(0,) \; _10))$$

From here

$$_1 = _0 + (,)[( ] -)$$

calling $_1 - _0$ the step size h, we get $_1 = _0 + (,)h$

One can now use the value of $_1$ (an approximate value of at y as x= $_1$) to calculate $_2$ and th given by

$_2 = _1 + (,)h.$

$_2 = _1 + h$

Based on the above equations, if we know the value of y= at ,then

$_{+1} = + hf(,)$

This formula is known as Euler's method and is illustrated graphically in Fig.A. It is also Called the Euler-Cauchy method.

A.b. **Euler's method based on problems**

1 .Using Euler's method, find y(0.2) and y(0.4) from $— = +$, y(0)=1with h=0.2.

Solution:

Given $— = +$, $_0 = 0, \; _0 = 1, h = 0.2$

We know that Euler's formula is

$$_{+1} = + hf(,)$$

$$= _0 + h$$

if n=0 ,

$$_1 = _0 + hf(_0, _0)$$

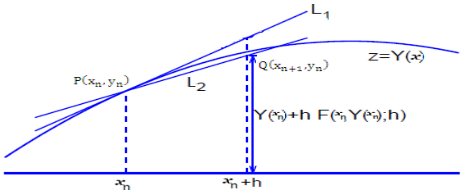
Where, $_1 =$ y(0.2)

$$=1+0.2 \; (\; _10 + \; _10)$$

$$=1+0.2(0+1)$$

$$=1+0.2$$

$$_1 =1.2$$



if n=1,

$$_2 = {}_1 + hf({}_1, {}_1)$$

$$=1.2+0.2({}_1 + {}_1)$$

$$=1.2+0.2(0.2+1.2)$$

$$=1.2+0.2(1.4)$$

$$=1.2+0.28$$

$$_2 \; =1.48$$

B. *Modified Euler's method*

We start with tangent line1 at ($_{\downarrow}0, {}_{\downarrow}0)$ .let the at $_0 + h\frac{}{2}$ intersect the tangent **Fig.B** Graphical interpretation of the first step Of modified Euler's method.Line2 at Q, Where y= $_0 + h\frac{}{2}\,(\; , \;)$.
We find the slope at that is

$$/ \;\; = ({}_{\downarrow}0 + h/2, \; {}_0 + (\; , \;) h\frac{}{}).$$

We draw the line through Q with this line2.we then draw the line L through $({}_0, {}_0)$parallel to line2 (L2) .this line L1 is taken as an approximation to the curve in the interval $({}_0, {}_1)$.

Fig.B Graphical interpretation of the first step Of modified Euler's method

Equation of L is

$$\frac{}{} = \frac{{}_1 - {}_0}{{}_1 - {}_0}$$

$$= ({}_{\downarrow}0 + h/2, \; {}_0 + h\frac{}{2}\,(\; , \;))$$

$$({}_{\downarrow}1 \; [- \;] \; {}_{\downarrow}0)/({}_{\downarrow}1 - {}_{\downarrow}0) = ({}_{\downarrow}0 + h/2, \; {}_0 + h\frac{}{2}\,(\; , \;))$$

$$_{\downarrow}1 \; [- \;] \; {}_{\downarrow}0 = ({}_{\downarrow}0 + h/2, \; {}_0 + h\frac{}{2}\,(\; , \;))({}_1 - {}_0)$$

When x= $_1 = {}_0$+h,

$$_{\downarrow}1 = {}_{\downarrow}0 + ({}_{\downarrow}0 + h/2, \; {}_0 + (\; , \;) h\frac{}{})({}_1 - {}_0)$$

i.e. $_1 = {}_0 + ({}_0 + h/2, {}_0 + (,) h\frac{}{}) h$

In general solution is

$$_{(+1)} = {}_{} + ({}_{} + h/2, \; + (,) h\frac{}{})] h$$, i=0, 1, 2, 3…

Where $= {}_0 + h$, $= ( ).$

B.a.Modified Euler's method based on problem

Solve $\frac{}{} = - {}^2 + 1, \; (0) = 0.5$ by modified Euler's method. Find the value y (0.2).

Solution:

Given

$\frac{}{} = - {}^2 + 1$, ${}_0 = 0, \; {}_0 = 0.5$

$\frac{}{} = - {}^2 + 1,$ Take h =0.1

Find: y (0.2)

By modified Euler's formula is

$$_{(+1)} = {}_{} + h\,[{}_{} + h/2, \; {}_{} + h/2 \; ({}_{(,)} \; {}_{})]$$, n=0, 1, 2, 3 … put n=0

$$_1 = {}_0 + h \; ({}_0 + h/2, \; {}_0 + (,) h\frac{}{})$$

$=0.5+0.1$ f $[0+ \frac{0.1}{2}, \; 0.5 + \frac{0.1}{2} \; (0,0.5)]$

$=0.5+0.1$ f $[0.05, \; 0.5 + 0.05(1 + 0.5)]$

$=0.5+0.1$ f $[0.05 , 0.575]$

$=0.5+0.1(0.575- [(0.575)]^2 + 1)$

$=0.65725$

${}_1 = 0.65725$

Now ${}_1 = 0.1, \; {}_1 = 0.65725$

i.e. y (0.1)= 0.65725

$$_2 = {}_1 + h \; ({}_1 + h/2, \; {}_1 + (,) h\frac{}{})$$

$= 0.65725 + 0.1 \; [0.1 + 0.05 , 0.65725 + 0.05(1.6425)]$

$= 0.65725 + 0.1 \; [0.15, 0.7396]$

$= 0.65725 + 0.1[0.7396 - [(0.15)]^2 + 1]$

$= 0.65725 + 0.17171$

${}_2 = 0.82896$

Y (0.2) = 0.83

C.Runge-kutta methods(R-K methods)

Runge – kutta methods are more accurate than the earlier methods we have seen .Two German mathematicians, Runge – kutta developed to solve a differential equation efficiently. The advantage of this method is that it requires only values of the function at some specified points. these methods agree with Taylors series expansion up to the terms of h ,where r is the order of the Range – kutta methods.The method two or more estimates of Δ ,the increment in y, are computed and a linear combination of these estimates are used to determine Δ and a linear combination of these estimates are used to determine Δ and hence the next value of y is

$$(\quad +h)=\quad (\quad)+\Delta\quad .$$

Since the derivations are complicate to solve

$$-=(\ ,\ )\ ,\ (\ _{0})=\ _{0}$$

By general Euler's method is

$$_{+1}=\quad +h(\ ,\ ),\ \text{n=0, 1, 2 …}$$

$$_{1}=hf(\ ,\ )$$

$$_{2}=hf\left(\quad +h\frac{\square}{2},\quad +\frac{_{1}}{2}\right),\ \Delta=\frac{1}{2(\ _{1}+\ _{2})}$$

$$\text{then}\quad _{+1}=\quad +\Delta\ ,\ \text{n=0, 1, 2…}$$

Note that Δ is the mean of $_{1}$ and $_{2}$ .some times Δ taken as $_{2}$, in which case second order Runge –kutta method is modified Euler's method.

Fourth –order algorithm is

$$_{1}=hf(\ ,\ )$$
$$_{2}=hf\left(\quad +h\frac{\square}{2},\quad +\frac{_{1}}{2}\right)$$
$$_{3}=hf\left(\quad +h\frac{\square}{2},\quad +\frac{_{2}}{2}\right)$$
$$_{4}=hf(\quad +h,\quad +\ _{3})$$

And $\Delta=\dfrac{1}{6(\ _{1}+2\ _{2}+2\ _{3}+\ _{4})}$

$$_{+1}=\quad +\Delta\ ,\quad \text{n=0, 1, 2…..}$$

Note that Δ is the mean of $_{1}$ $_{2}$, $_{3}$ and $_{4}$.

$$\text{i.e.,}\quad _{+1}=\quad +\frac{1}{6(\ _{1}+2\ _{2}+2\ _{3}+\ _{4})},\quad \text{n=0,1,2…..}$$

C. RUNGE-KUTTA METHOD BASED ON PROBLEM

1. Using Runge-kutta method of fourth order solve $-=\dfrac{^{2}-\ ^{2}}{+\ ^{2}}$ with $(0)=1\qquad =0.2.$

Solution:

Given: f(x, y) = $\dfrac{^{2}-\ ^{2}}{^{2}+\ ^{2}}$, $_{0}=0,\ _{0}=1,h=0.2$

The Fourth –order Runge-kutta method is $_{1}=h(\ ,\ )$

$$_{\downarrow}2=\llbracket h(\ \rrbracket\ _{\downarrow}+h/2,\quad +\frac{_{1}}{2})$$
$$_{\downarrow}3=\llbracket h(\ \rrbracket\ _{\downarrow}+h/2,\quad +\frac{_{2}}{2})$$
$$_{\downarrow}4=\llbracket h(\ \rrbracket\ _{\downarrow}+h,\quad +\ _{3}),\ \text{n=0, 1, 2…}$$

Put n=o,

$$_{1}=h(\ _{0},\ _{0})$$

$$=h\left(\frac{_{1}{}^{2}-\ _{1}{}^{2}}{_{1}{}^{2}+\ _{1}{}^{2}}\right)$$

$$=0.2\left(\frac{1-0}{1+0}\right)$$

$_{1}$ =0.2

$${}_{\downarrow}2 = \llbracket h\,(\;\rrbracket\;{}_{\downarrow}0 + h/2,\;{}_{0} + \frac{1}{2})$$

=0.2f (0.1, 1+0.1)

=0.2f (0.1, 1.1)

$$=0.2\left(\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right)$$

$$=0.2\left(\frac{1.20}{1.22}\right)$$

$_{2}$ =0.1967

$${}_{\downarrow}3 = \llbracket h\,(\;\rrbracket\;{}_{\downarrow}0 + h/2,\;{}_{0} + \frac{2}{2})$$

=0.2f (0.1, 1+0.09835)

=0.2f (0.1, 1.09835)

$$=0.2\left(\frac{(1.09835)^2 - (0.1)^2}{(1.09835)^2 + (0.1)^2}\right)$$

$$=0.2\left(\frac{1.2046 - 0.01}{1.2046 + 0.01}\right)$$

$$=0.2\left(\frac{1.1964}{1.2164}\right)$$

=0.1967

$${}_{\downarrow}4 = \llbracket h\,(\;\rrbracket\;{}_{\downarrow}0 + h,\;{}_{0} + \;{}_{3}),$$

=0.2f (0.2, 1+0.197)

=0.2f (0.2, 1.197)

$$=0.2\left(\frac{(1.197)^2 - (0.2)^2}{(1.197)^2 + (0.2)^2}\right)$$

$$=0.2\left(\frac{1.4321 - 0.04}{1.4321 + 0.04}\right)$$

$$=0.2\left(\frac{1.3921}{1.4721}\right)$$

$_{4}$ =0.1891

$$\Delta \; = \frac{1}{6(_1 + 2 \; _2 + 2 \; _3 + \; _4)}$$

$$\Delta \; = \frac{1}{6(0.2 + 2(0.1967) + 2(0.1967) + 0.1891)}$$

$$\Delta \quad = 0.196$$

$$_1 = \; _0 + \Delta$$

$$=1+0.196$$

$$_1 \quad =1.196$$

Put n=1,
To find $_2$

$$_1(0.2) \quad =1.196, \quad _1 = 0.2, \quad _1 = 1.196$$

$$_1 = h\,(_1, \; _1)$$

$$=h\left(\frac{_1{}^2 - \; _1{}^2}{_1{}^2 + \; _1{}^2}\right)$$

$$=0.2\left(\frac{(1.196)^2 - (0.2)^2}{(1.196)^2 + (0.2)^2}\right)$$

$$_1 =0.2(\frac{1.43 - 0.04}{1.43 + 0.04})$$

$$_1 =0.2(\frac{1.39}{1.47})$$

$$_1 =0.1891.$$

$$_12 = [\![h \; (\;]\!] \; _11 + h/2, \quad _1 + \frac{1}{2})$$

$$=0.2f\ (0.3,\ 1.296)$$

$$=0.2\left(\frac{(1.296)^2 - (0.3)^2}{(1.296)^2 + (0.3)^2}\right)$$

$$=0.2\left(\frac{1.587}{1.767}\right)$$

$$_2 =0.171$$

$$_13 = [\![h \; (\;]\!] \; _11 + h/2, \quad _1 + \frac{2}{2})$$

$$=0.2f\ (0.3,1.282\ )$$

$$=0.2\left(\frac{(1.282)^2-(0.3)^2}{(1.282)^2+(0.3)^2}\right)$$

$$=0.2\left(\frac{1.554}{1.734}\right)$$

$$=0.2(0.896)$$

$$_3=0.179$$

$$_{1}4 = [h\ (\ ]\ _{1}1 + h,\ _1 + \ _3),$$
$$=0.2f\ (0.3,\ 1.375)$$

$$=0.2\left(\frac{(1.375)^2-(0.3)^2}{(1.375)^2+(0.3)^2}\right)$$

$$=0.2\left(\frac{1.801}{1.981}\right)$$

$$_4=0.1818$$

$$\Delta = \frac{1}{6(\ _1+2\ _2+2\ _3+\ _4)}$$

$$=\frac{1}{6(0.1891+2(0.171)+2(0.179)+0.1818)}=0.178$$

$$_2 = \ _1+\Delta$$
$$=1.196+0.178$$
$$_2=1.374$$

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