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Application of Finite Element Method in Electromagnetic Field Calculation

Rajat Shubhra Pal
Electrical Engineering Department
Institute of Engineering & Management
Kolkata, India
Email:rajat.electengg@gmail.com

Abstract: **This paper discusses the necessity of finite element method in solving electromagnetic field problems. It also explains that in some cases finite element method can show greater accuracy than normal analytical method. For example to calculate the capacitance of a parallel plate capacitor normal analytical method neglect fringing effect whereas finite element method shows that considering fringing effect more accurate value can be obtained.**

Keywords: Electromagnetic Field, Finite Element Method, Helmholtz Equations.

I. INTRODUCTION

The basic task of field theory is to find out electric or magnetic field (or both) for different types of charge distribution or current distribution (or both). Various laws and techniques are available in field theory to calculate electric and magnetic field. For example in electrostatics there is Coulamb's law, Gauss's law; in magnetostatic there is Bio-savart law, Ampere's law. However, they have their own limitations. Probably the best way is to find out the potential field first and then the field. The governing equations to find out the potentials are Poission's and Laplaces's equations (for static fields) and Helmholtz's equation (for time varying field). All these equations are in partial differential form which required rigorous analytical calculation. However, analytical solutions are available only for problems with simple configurations. Several real-world electromagnetic problems are not analytically calculable due to the irregular geometries found in actual devices.

When the complexities of theoretical formulas make analytic solution inflexible, one has to rely on nonanalytic methods, like (1) graphical methods, (2) experimental methods, (3) analog methods, and (4) numerical methods. Graphical, experimental, and analog methods are applicable to solving relatively few problems. Numerical methods have come into importance and become more attractive with the arrival of fast digital computers. The most widely used numerical method is the finite element method (FEM), which can be used in the analysis of two- or three-dimensional electromagnetic field problems. The solution can be obtained for static, time-harmonic or transient problems.

II. ELECTROMAGNETIC PRINCIPLES

Maxwell's equations are written in differential form as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathrm{B}}{\partial t} \quad (1)$$

$$\nabla \times \mathbf{H} = \mathrm{J} + \frac{\partial \mathrm{D}}{\partial t} \quad (2)$$

$$\nabla \cdot \boldsymbol{D} = \rho, \quad (3)$$

$$\nabla \cdot \boldsymbol{B} = 0. \quad (4)$$

The magnetic vector potential $\boldsymbol{A}$ is given by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \qquad (5)$$

The substitution of the definition for the magnetic vector potential in equation (1) yields

$$\nabla \times \left(\boldsymbol{E} + \frac{\partial A}{\partial t}\right) = 0 \qquad (6)$$

Applying the vector identity that Curl of Grad of any scalar is zero in equation (6) one gets

$$\mathbf{E} = -\frac{\partial \mathrm{A}}{\partial \mathrm{t}} - \nabla \mathrm{V} \qquad (7)$$

where V is the reduced electric scalar potential. The equation (7) shows that the electric field strength vector consists of two parts, namely a rotational part induced by the time dependence of the magnetic field, and an irrotational part created by electric charges and the polarization of dielectric materials.

Again from equation (1) using the relation $\mathbf{D} = \varepsilon\mathbf{E}$ and incorporating equation (7)

$$-\frac{\partial}{\partial t}(\nabla.\boldsymbol{A}) - \nabla^2 V = \frac{\rho_v}{\epsilon} \qquad (8)$$

Taking curl on both side of equation (2) and incorporating equation (5) and (7) one can get after simplification

$$\nabla^2 \boldsymbol{A} - \nabla(\nabla.\boldsymbol{A}) = -\mu \boldsymbol{J} + \mu\epsilon\nabla\left(\frac{\partial V}{\partial t}\right) + \mu\epsilon\frac{\partial^2 A}{\partial t^2} \qquad (9)$$

Applying *Lorentz condition of potential* [1]

$$\nabla.\boldsymbol{A} = \mu\epsilon\left(\frac{\partial V}{\partial t}\right) \qquad (10)$$

in equation (8) and (9) one can obtain

$$\nabla^2\varphi - \mu\epsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\epsilon} \qquad (11)$$

$$\nabla^2\boldsymbol{A} - \mu\epsilon\frac{\partial^2 A}{\partial t^2} = -\mu\boldsymbol{J} \qquad (12)$$

These are *Helmholtz's equations*, and for static case these are reduces to well known Poission's (or Laplace's) equations [2]

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \qquad (13)$$

$$\nabla^2\boldsymbol{A} = -\mu\boldsymbol{J} \qquad (14)$$

Further, there are two types of boundary conditions. Dirichlet's boundary condition indicates that a known potential, here the known vector potential

$$\mathbf{A} = \text{constant}, \qquad (15)$$

can be achieved for a vector potential for instance the field is parallel to the contour of the surface. Neumann's homogeneous boundary condition determined with the vector potential

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$$\nu \frac{\partial A}{\partial n} = 0 \tag{16}$$

can be achieved when the field meets a contour perpendicularly. Here ***n*** is the normal unit vector of a plane.

III. FINITE ELEMENT METHOD

The finite element analysis of any problem involves basically four steps: (a) discretizing the solution region into a finite number of subregions or elements, (b) deriving governing equations for a typical element, (c) assembling of all elements in the solution region, and (d) solving the system of equations obtained.

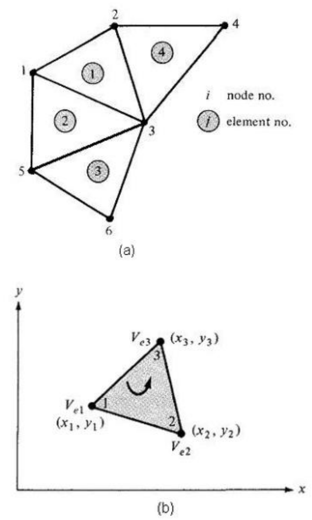

Fig 1- Finite element solution pattern

The solution region is divided into a number of finite elements as illustrated in Figure 1(a) where the region is subdivided into four non-overlapping elements and six nodes. An approximation for the potential V_e within an element e is required and then the potential distributions in various elements are to be interrelated such that the potential is continuous across inter-element boundaries. The most common and easiest form of approximation for V_e for a triangular element is polynomial approximation [4],

$$V_e(x, y) = a + bx + cy \tag{17}$$

Consider a triangular element shown in Figure 2(b). The potential V_{e1}, V_{e2}, and V_{e3} at nodes 1, 2, and 3, respectively, are obtained using eq. (8); that is in matrix form

$$\begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{18}$$

Calculating the coefficients a, b, and c from eq. (18) and substituting this into eq. (17) gives

$$V_e = \sum_{i=1}^{3} \alpha_i(x, y) V_{ei} \tag{19}$$

Where α_i are called the element shape functions.

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The energy per unit length associated with the element e is given by [3]

$$W_e = \frac{1}{2}\int \varepsilon |E|^2 ds = \frac{1}{2}\int \varepsilon |\nabla V_e|^2 ds \quad (20)$$

Calculating grad V_e from eq (19) and substituting it into eq (20) in matrix form one can write

$$W_e = \frac{1}{2}\varepsilon [V_e]^T [C^{(e)}][V_e] \quad (21)$$

The matrix $C^{(e)}$ is usually called the element coefficient matrix.

Assembling all elements in the solution region one has,

$$W = \sum_{e=1}^{N} W_e = \frac{1}{2}\varepsilon [V]^T [C][V] \quad (22)$$

[C] is called the overall or global coefficient matrix.

From variational calculus [5], it is known that Laplace's (or Poisson's) equation is satisfied when the total energy in the solution region is minimum. Thus it is required that the partial derivatives of W with respect to each nodal value of the potential be zero; that is,

$$\frac{\partial W}{\partial V_1} = \frac{\partial W}{\partial V_2} = \cdots = \frac{\partial W}{\partial V_N} = 0 \quad (23)$$

After differentiating equation (22) as described in [3], one will get a set of matrix of the form

$$[A][V] = [B] \quad (24)$$

Thus the potentials at the free nodes can be obtained from matrix [V] as

$$[V] = [A]^{-1}[B] \quad (25)$$

IV. STATEMENT OF THE PROBLEM AND SOLUTIONS

Consider the parallel-plate capacitor of Figure 2. Suppose that each of the plates has an area $A=0.03\ m^2$ and they are separated by a distance $d=0.02\ m$.

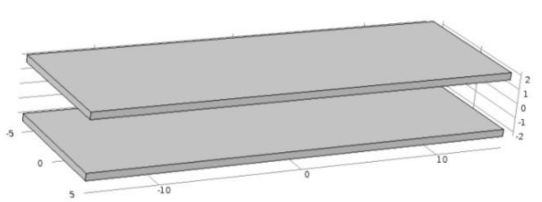


Fig 2- Parallel plate capacitor

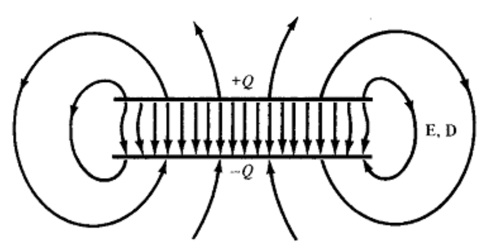


Fig 3- fringing effect due to a parallel-plate capacitor

Applying a voltage difference results in an electric field that extends not just directly between the conductive plates, but extends some distance away, a phenomenon referred to as a fringing field as illustrated in Figure 3.

A. *Analytical Method*

In normal analytical method one has to assume an ideal parallel-plate capacitor, in which the plate separation is very small compared with the dimensions of the plate. Assuming such an ideal case, the fringing field at the edge of the plates can be ignored so that the field between them is considered uniform. If the space between the plates is filled with a homogeneous dielectric with permittivity ε_0 and the parallel plates in Fig. 2 are maintained at a potential difference V_0 so that $V(z = 0) = 0$ and $V(z = d) = V_0$, from eq. (13) one can write,

$$\nabla^2 V = \frac{\partial^2 V}{\partial z^2} = 0 \qquad (26)$$

And solving this equation with the boundary conditions it can be obtained

$$V = \frac{V_0}{d} z \qquad (27)$$

$$\text{And } \boldsymbol{E} = -\frac{\partial V}{\partial z}\boldsymbol{a_z} = -\frac{V_0}{d}\boldsymbol{a_z} \qquad (28)$$

From equation (3) surface charge density on conducting plates

$$\rho_s = \nabla.\boldsymbol{D} = \varepsilon_0(\nabla.\boldsymbol{E}) = -\frac{\varepsilon_0 V_0}{d} \qquad (29)$$

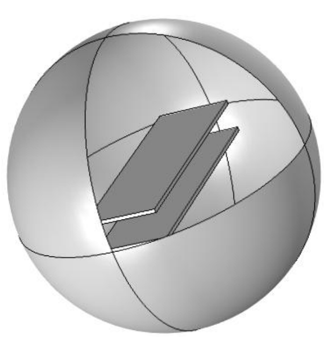
Now capacitance

$$C = \frac{Q}{V} = \frac{\rho_s A}{V_0} = \frac{\varepsilon_0 A}{d} = 13.28 \times 10^{-12}\ \text{F} \qquad (30)$$

Clearly the above mention method is not accurate as the fringing effect is neglected. To accurately predict the capacitance of a capacitor, it is necessary to use a modeling domain sufficiently large to include this fringing field and to use appropriate boundary conditions.

B. Finite Element Method

Fig 4 shows the capacitor consisting of two metal plates in an imaginary spherical volume of air. The size of the sphere defines the modeling space. This model studies the various sizes of this imaginary sphere and its effect upon the capacitance.



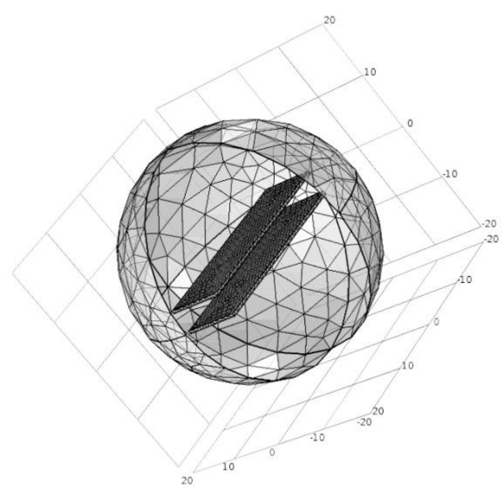


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Fig 4- An imaginary sphere

Fig 5 - mesh generation

The imaginary sphere boundary can be thought of as one of two different physical situations:

(1) It can be treated as a perfectly insulating surface, across which charge cannot redistribute itself, or (2) as a perfectly conducting surface, over which the potential will not vary.

The modeling realization of the perfectly insulating surface is the Zero Charge boundary condition where the electric field lines are tangential to the boundary. The modeling realization of the perfectly conducting surface fixes the voltage of all of the boundaries of the sphere to a constant. The boundary condition also implies that the electric field lines are perpendicular to the boundary.

V. RESULTS

Radius of the imaginary sphere is chosen as 20cm to 30cm at 2cm interval. For each radius capacitances are calculated once for insulating sphere and then for conducting sphere.

Table 1 shows the values of capacitance (in pF) obtained from finite element calculation for insulating imaginary sphere and for conducting imaginary sphere.

Table I. Capacitances obtained for different sphere radius

Radius of the imaginary sphere (cm)	Capacitance (pF) for insulating sphere	Capacitance (pF) for conducting sphere
20	12.556301865011	13.6243831021679
22	12.7445295807392	13.4757171271170
24	12.8732449244067	13.4099074411108
26	12.9755203765344	13.3864597253743
28	13.037399291021	13.3600241674733
30	13.082748216695	13.3419305910274

Fig 6 compares the capacitance values of the device with respect to imaginary sphere radius for the two boundary conditions. The figure also plots the average of the two values. Notice that all three capacitance calculations converge to the same value as the radius grows. It is also interesting that average value is very clos to the analytical value 13.28 pF.

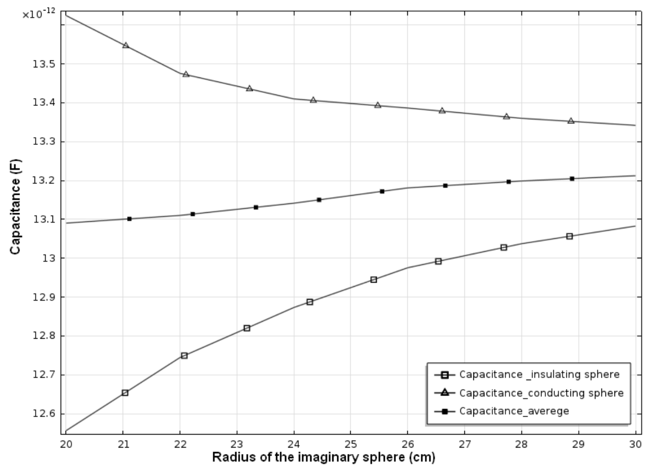

Fig 6- variation of capacitance with the increase of air sphere radius

VI. CONCLUSION

The above example shows that where classical analytical methods consider lots of assumptions and produces approximate results, finite element method can handle more parameters and produce more accurate results. Classical method suitable for lumped parameter representation, where finite element can take care of minute changes in the field so it is better suitable for distributed representation. Besides, sometimes equations (11) and (12) become unsolvable using normal analytical method. In that situation one has to rely on numerical method like finite element or finite difference method [7].

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